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On Singular Generalized Absolutely Monotone Functions

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Bounded generalized absolutely monotone functions which are not equal to their Taylor-type series are considered. This family of functions constitutes a convex cone in a generalized $C^\infty(a, b)$ space. The question of extreme rays of this cone as well as the extreme ray representation of its elements is discussed. © 1994 Academic Press, Inc.

1. INTRODUCTION

We start by recalling some definitions and results to be used in the sequel. Let $\{u_i\}_{i=0}^\infty$ be an infinite sequence of functions belonging to $C^\infty[a, b]$, such that for all n, $n = 0, 1, 2, \ldots$, $\{u_i\}_{i=0}^n$ forms an Extended Tchebycheff System on $[a, b]$. With no loss of generality we may assume that

$$u_i(t) = \phi_i(t; a), \qquad i = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$\phi_0(t; x) = \begin{cases} 0, & a \le t < x, \\ w_0(t), & x \le t \le b, \end{cases} \tag{1.2}$$

$$\phi_i(t; x) = \begin{cases} 0, & a \le t < x, \\ \int_x^t w_i(\xi) \phi_{i-1}(t; \xi)\, d\xi), & x \le t \le b, \end{cases} \tag{1.3}$$

$$i = 1, 2, 3, \ldots,$$

and where $\{w_i\}_{i=0}^\infty$ is a sequence of positive $C^\infty[a, b]$ functions.

1.1. DEFINITION. A function f defined on (a, b) is said to be convex with respect to the Tchebycheff system $\{u_i\}_{i=0}^n$ if for every set of $n + 2$ points, $a < t_0 < t_1 < \cdots < t_{n+1} < b$, the following determinantal

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inequality holds:

$$U\begin{pmatrix} u_0, u_1, \ldots, u_n, f \\ t_0, t_1, \ldots, t_n, t_{n+1} \end{pmatrix} = \begin{vmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) & u_0(t_{n+1}) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) & u_1(t_{n+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) & u_n(t_{n+1}) \\ f(t_0) & f(t_1) & \cdots & f(t_n) & f(t_{n+1}) \end{vmatrix} \geq 0.$$

The set of convex functions with respect to the Tchebycheff system $\{u_i\}_{i=0}^n$ forms a convex cone denoted by $C(u_0, u_1, \ldots, u_n)$ or C_n in case no ambiguity arises. Also, we let C_{-1} denote the cone of nonnegative functions on (a, b). Note that $\phi_k(\cdot; x)$, for $k \geq n$, is in C_n (see [2, Coro. 3.2, p. 395]).

It is shown in [2] that $f \in C_A = \cap_{n=-1}^{\infty} C_n$ if and only if

$$(L_{-1}f)(t) = f(t) \geq 0$$

and

$$(L_n f)(t) = (D_n D_{n-1} \cdots D_0 f)(t) \geq 0, \qquad a < t < b, n = 0, 1, 2, \ldots$$

where $(D_k f)(t) = (d/dt)(f(t)/w_k(t))$.

The elements of the cone C_A are called generalized absolutely monotone (GAM) functions.

Also, if $f \in C_A$ then the following Taylor-type formulae hold (see [2, Remark 3.1, p. 395]):

$$f(t) = \int_a^b \phi_n(t; x)(L_n f)(x)\, dx + \sum_{i=0}^{n} \frac{(L_{i-1} f)(a+)}{w_i(a)} u_i(t), \tag{1.4}$$
$$a \leq t < b, \qquad n = 0, 1, 2, \ldots .$$

Formulae (1.4) give extreme ray representations for the elements of $\cap_{i=-1}^{n} C_i$.

As shown in [1], a necessary and sufficient condition for all functions $f \in C_A$ to admit the Taylor-type representation

$$f(t) = \sum_{i=0}^{\infty} a_i u_i(t), \tag{1.5}$$

where

$$a_i = \frac{(L_{i-1} f)(a+)}{w_i(a)}, \qquad i = 0, 1, 2, \ldots,$$

is that for every t, $a < t < b$, there exists a number s, $t < s < b$, such that

$$\lim_{i \to \infty} u_i(t)/u_i(s) = 0. \tag{1.6}$$

Moreover if we restrict ourselves to the cone $B \cap C_A$, where B denotes the set of bounded functions on (a, b) then (1.6) could be replaced by

$$\lim_{i \to \infty} u_i(t)/u_i(b) = 0. \tag{1.6'}$$

Formula (1.5) is an extreme ray representation for $f \in C_A$. In this paper we generalize the representation (1.5) for $B \cap C_A$-functions in case (1.6′) does not hold.

We conclude this section with:

1.2. LEMMA. *Let $m > n > 0$ and $a \leq y \leq x < b$ be fixed. The equation (in t)*

$$\frac{\phi_m(t; y)}{\phi_m(b; y)} - \frac{\phi_n(t; x)}{\phi_n(b; x)} = 0 \tag{1.7}$$

has at most one root in the interval (y, b). Moreover, if it has a root in this interval, then the left-hand side of (1.7) changes sign at this root.

Proof. Assume to the contrary that (1.7) has more than one root. Let $t_1 < t_2$ be two roots of (1.7) in (y, b). Clearly, t_1 and t_2 belong to (x, b). Define

$$f = \phi_m(\cdot; y)/\phi_m(b; y) - \phi_n(\cdot; x)/\phi_n(b; x).$$

Assume first that $n = 1$,

$$f|(x, b) \in C(u_0|[x, b], u_1|[x, b]), \tag{1.8}$$

where $g|J$ denotes the restriction of g to the set J. Since f vanishes at the points t_1, t_2 and b, f agrees with a "*polynomial*" $a_0u_0 + a_1u_1$ on $[t_1, b]$ (see [4, Lemma 1]). This is impossible by the definition of $\phi_m(\cdot; x)$ and since $m > 1$.

Suppose f has a single zero, t_0, in (y, b) and that f does not change sign at this point. If $y < x$ then $f(x) > 0$, and, since f doesn't change sign, it is strictly positive in (t_0, b). In this case,

$$U\begin{pmatrix} u_0, u_1, f \\ t_0, t, b - \end{pmatrix} < 0 \tag{1.9}$$

for all $t \in (t_0, b)$, in contradiction to (1.8). If $y = x$ then, since

$\{\phi_i(t; y)/\phi_i(b; y)\}_{i=0}^{\infty}$ is a nonincreasing sequence for all fixed t and y (see [1, Lemma]), $f \leq 0$. It follows that

$$U\begin{pmatrix} u_0, u_1, f \\ t_1, t_0, t_2 \end{pmatrix} < 0, \tag{1.10}$$

for every $t_1 \in (x, t_0)$ and $t_2 \in (t_0, b)$. Inequality (1.10) contradicts (1.8).

Let $n > 1$. Since $f(y) = 0$, f has at least four zeros in $[y, b]$. Hence $L_0 f$ has at least three zeros in the interval (y, b) (actually in (x, b)). The claim follows by induction since $f(y) = L_0 f(y) = L_1 f(y) = \cdots = L_{n-2} f(y) = 0$, and at each stage $L_k f$ has at least three zeros in (x, b).

The proof that f cannot vanish at a single point of (y, b), without changing sign at that point, follows in the same lines. ∎

In what follows we assume, for the sake of simplicity, that $w_0 = 1$ (which implies that the elements of C_0 are nondecreasing in (a, b)).

2. THE CONE OF SINGULAR GAM FUNCTIONS

2.1. DEFINITION. A function f which (i) belongs to C_A (or $B \cap C_A$) and (ii) satisfies $((L_i f)/w_{i+1})(a+) = 0$ for $i = -1, 0, 1, \ldots,$ is called a singular generalized absolutely monotone (SGAM) function.

2.2. COROLLARY. *Assuming that* (1.6) (*resp.* (1.6′)) *holds, then the only singular function in* C_A (*resp.* $B \cap C_A$) *is the zero function.*

In [7], Ziegler raises the question of the extreme ray structure of C_A in case (1.6) does not hold. In [3], we gave an example of an infinite sequence defined by (1.1)–(1.3) for which (1.6′) does not hold. In this note we discuss the extreme ray structure of the cone $B \cap C_A$ in case that (1.6′) does not necessarily hold, and find an extreme ray representation for its elements when certain conditions are satisfied.

Since every GAM function has a unique representation

$$f = f_1 + f_0, \tag{2.1}$$

where

$$f_1 = \sum_{i=0}^{\infty} \frac{((L_{i-1} f)(a+))}{w_i(a)} u_i \tag{2.2}$$

and f_0 an SGAM function, it is sufficient to discuss the extreme ray representation of SGAM functions.

The set of the SGAM functions is a convex cone with vertex at the origin. This cone will be denoted by S. The cone S, as well as C_A, are subsets of the generalized $C^\infty(a, b)$ space V, i.e., the linear space of the functions for which the differential operators L_i, $i = -1, 0, 1, \dots$ are defined, with the topology determined by the family of seminorms,

$$\|f\|_k^n = \sup\{|L_p f(t)| \,|\, t \in I_k, p \le n\}, \tag{2.3}$$

where $I_k = [a + (1/k), b - (1/k)]$, $k > 2/(b - a)$ and $n = -1, 0, 1, \dots$. With this topology, V is a complete metrizable locally convex space. Moreover, it is also a Montel space, i.e., every bounded set is relatively compact (see [1]). In particular, the set $\{f \in C_A | \lim_{t \to b} f(t) \le 1\}$ is closed and bounded, hence compact [1].

2.3. LEMMA. *The limit $\phi(t; x) = \lim_{n \to \infty} \phi_n(t; x)/\phi_n(b; x)$ exists for every $t \in [a, b]$ and $x \in [a, b)$. Moreover, it has the following properties:* (i) *for every x, $\phi_x = \phi(\cdot\,; x) \in S$,* (ii) *for every t, $\phi^t = \phi(t; \cdot\,)$ is nonincreasing, and* (iii) *ϕ^t is left-continuous.*

Proof. The functions $\phi_n(t; \cdot\,)/\phi_n(b; \cdot\,)$, $n = 0, 1, 2, \dots$, are continuous and nonnegative. Moreover, they are nonincreasing [2, Lemma 9.2, p. 437]. This, together with the fact that $\{\phi_n(t; x)/\phi_n(b; x)\}_{n=0}^\infty$ is a nonincreasing sequence for every fixed t and x (see [1, Lemma]), implies the existence of the limit as well as (ii) and (iii).

Since $(\phi_n(\cdot\,; x)/\phi_n(b; x)) \in C_m$ for all $n \ge m$ and $0 \le (\phi_n(\cdot\,; x)/\phi_n(b; x)) \le 1$ and since C_m is closed under pointwise convergence, it follows that $\phi_x \in B \cap C_A$. Since for $x > a$ and for all n, $\phi_n(\cdot\,; x)$ vanishes on $[a, x]$, so does ϕ_x. This implies that ϕ_x is singular. For the case $x = a$, see [1]. ∎

For $t \in [a, b]$ and $x \in [a, b)$, define $\psi_n(t; x) = (\phi_n(t; x)/\phi_n(b; x))$. Since $\psi_n(t; \cdot\,)$ is nonincreasing and bounded one can define $\psi_n(t; b) = \lim_{x \to b} \psi_n(t; x)$. Applying L'Hospital's rule one sees that $\psi_n(t; b) = 0$ for $a \le t < b$ and hence $\lim_{t \to b} \psi_n(t; b) = 0$, however, $\psi_n(b; b) = 1$. The functions $\psi_n(t; \cdot\,)$ are continuous on $[a, b]$.

2.4. COROLLARY. *For every $t \in (a, b)$, the closed set* $\operatorname{supp}(\phi(t; \cdot\,))$ *is either empty (in case that (1.6′) holds, these sets are empty for all t) or a closed interval $[a, a_t]$, for some $a_t \ge a$.*

2.5. LEMMA. *Let f be a bounded SGAM function. Then*

$$f(t) = \int_a^b \psi_n(t; x)\, d\alpha_n(x), \tag{2.4}$$

where

$$\alpha_n(x) = \int_a^x \phi_n(b;\xi)(L_n f)(\xi)\,d\xi. \tag{2.5}$$

Proof. The proof follows from (1.4) and the fact that f is singular. Moreover, α_n is continuous and nondecreasing on $[a, b]$. ∎

We use the following notation: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and let $M = \{n_j\}_{j=1}^{\infty}$ be a subsequence of integers. Then $M - \lim x_n$ denotes $\lim_{j\to\infty} x_{n_j}$.

2.6. LEMMA. *Let $f \in S$ and let the functions α_n be defined by* (2.5). *Then there exists a function α and a sequence $M(\alpha) = \{n_j\}_{j=1}^{\infty}$ such that*

$$\alpha(x) = M(\alpha) - \lim \alpha_n(x)$$

exists for every x.

Proof. For every n, α_n is a positive nondecreasing function and $\alpha_n(x)$ is bounded by $f(b-)$. The claim follows by Theorem 16.2 of [6, p. 27]. ∎

2.7. DEFINITION. Let the function α be nondecreasing (nonincreasing) in $I = [a, b]$. A point $x \in I$ is a point of invariability of α if α is constant in some neighborhood of x. All the other points are called points of increase (resp. decrease) (see [6, p. 6]).

2.8. DEFINITION. Let f be an element of a cone C whose vertex is at the origin. We say that f generates an extreme ray in C if $\rho = \{rf \mid r \geq 0\}$ is an extreme subset of C. In this case ρ is called an extreme ray of C.

2.9. THEOREM. *Let f be a nonzero SGAM function and let α_n and α be defined as in Lemmas* 2.5 *and* 2.6. *If α has more than one point of increase then f does not generate an extreme ray of S.*

Proof. Since f is not identically equal to zero, we may assume that $f(b-) = 1$. Since $\psi_n(b; x) = 1$ for every x, (2.4) implies that $\alpha_n([a, b])$ and $\alpha([a, b])$ are both equal to 1, where $\alpha_n(J) = \int_J d\alpha_n$ and $\alpha_n(J) = \int_J d\alpha_n$ for every measurable set J. Define the set

$$A = \{x \mid \phi(t;x) > 0, \quad \text{for some } t \in (a,b)\} = \{x \mid \phi(b-;x) > 0\},$$

and let $s = \sup A$. Clearly, $a \leq s \leq b$. Note that A is an interval ($[a, s)$ or $[a, s]$), since for each t, $\phi(t; x)$ is a nonincreasing function of x. First we show that α does not have points of increase in $(s, b]$. If $s = b$ then there is nothing to prove. Assume that $a \le s < b$. For every $t \in [a, b)$,

$$f(t) = \int_a^b \psi_n(t; x)\, d\alpha_n(x) = \int_a^{s_1} \psi_n(t; x)\, d\alpha_n(x) + \int_{s_1}^b \psi_n(t; x)\, d\alpha_n(x),$$

for all s_1, $s < s_1 < b$. Since for $x > s$, $\lim_{n\to\infty} \psi_n(t; x) = 0$, and for every n the function $\psi_n(t; x)$ decreases in x, then for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for all $n > n(\varepsilon)$,

$$f(t) \le \int_a^{s_1} \psi_n(t; x)\, d\alpha_n(x) + \varepsilon. \tag{2.6}$$

Letting $t \to b-$, we have, $1 = f(b-) \le \alpha_n([a, s_1]) + \varepsilon$. Letting $n \to \infty$, we get $1 \le \alpha([a, s_1]) + \varepsilon$. Since this holds for all ε and all s_1, $s < s_1 < b$, we have $\alpha([a, s]) = 1$. If s is not in A then $\psi_n(t, x) = 0$ for all $x \ge s$. This, together with the monotonicity and the continuity of $\psi_n(t, x)$ in x, implies that (2.6) holds with some $s_1 = s_1(\varepsilon)$, $a < s_1 < s$ and for all large n. Similar argument leads to the conclusion that $\alpha([a, s)) = 1$. In any case, $\alpha(A) = 1$. In particular, α does not have points of increase in $(s, b]$. Moreover, if s is not in A and is a point of increase of α then every neighborhood of s contains infinitely many points of increase of α.

Suppose α has at least two points of increase. Let c lie between two points of increase. Set

$$\beta_n(x) = \begin{cases} \alpha_n(x), & a \le x \le c, \\ \alpha_n(c), & c < x \le b, \end{cases}$$

$$\gamma_n(x) = \begin{cases} 0, & a \le x \le c, \\ \alpha_n(x) - \alpha_n(c), & c < x \le b. \end{cases}$$

Now define the functions g_n and h_n by

$$g_n(t) = \int_a^b \psi_n(t; x)\, d\beta_n(x),$$

and

$$h_n(t) = \int_a^b \psi_n(t; x)\, d\gamma_n(x).$$

Since $g_n + h_n = f$ and $f \in \cap_{i=-1}^n C_i$,

$$g_n, h_n \in \cap_{i=-1}^n C_i,$$
$$g_n \le f(b-),$$

and

$$h_n \leq f(b-).$$

As in Lemma 2.6, there exist two subsequences $M(g)$ and $M(h)$ such that

$$M(g) = -\lim g_n(t) = g(t)$$

and

$$M(h) - \lim h_n(t) = h(t).$$

We may assume that $M(\alpha) = M(g) = M(h)$. Clearly $g, h \in S$ and $f = g + h$. It is readily seen that $h = 0$ on $[a, c]$ while f, hence g, does not vanish on this interval. Also, since for some $t \in (a, b)$, $\phi(t; \cdot)$ is positive in an interval entirely to the right of c and containing a point of increase of α, $h \neq 0$ on (a, b). Thus g and h do not belong to the same ray of S, so f does not generate an extreme ray of S. ∎

In what follows we study the structure of $\phi(t; x)$ and give a representation of f by means of a certain set containing $\{\phi_x | a \leq x < b\}$. Assume first that ϕ^t is continuous for some t. In this case, Dini's Theorem implies that the convergence of $\psi_n(t; x)$ to $\phi^t(x)$ is uniform in x. Letting n go to infinity, (2.4) implies

$$f(t) = \int_a^b \phi^t(x)\, d\alpha(x) = \int_a^b \phi_x(t)\, d\alpha(x).$$

We now discuss the discontinuities of the functions $\{\phi^t\}$. If for some t, $\phi^t(x) \neq \phi^t(x+) = \lim_{y \to x+} \phi^t(y)$ then $\phi_x \neq \phi_{x+}$, where ϕ_{x+} is defined by $\phi_{x+}(s) = \phi^s(x+)$, $a \leq s \leq b$. We show that the discontinuities of $\phi(t;x)$ occur along segments.

2.10. LEMMA. *Let $\phi_x(t) \neq \phi_{x+}(t)$ for some t. If $s < t$ and $\phi_x(s) > 0$ then $\phi_x(s) \neq \phi_{x+}(s)$.*

Proof. For every t, set $X^t = \{x | \phi_x(t) \neq \phi_{x+}(t)\}$. We show that if $x \in X^t$ for some t then $x \in X^s$ for every $s < t$ as long as $\phi_x(s) > 0$. By [2, Lemma 9.2, p. 437], we have

$$\begin{vmatrix} \phi_n(s_1; x) & \phi_n(s_2; x) \\ \phi_n(s_1; y) & \phi_n(s_2; y) \end{vmatrix} \geq 0 \tag{2.7}$$

for $s_1 < s_2$ and $x < y$.

Dividing the rows of (2.7) by $\phi_n(b; x)$ and $\phi_n(b; y)$, respectively, we can write it in the form

$$\begin{vmatrix} \psi_n(s_1; x) & \psi_n(s_2; x) \\ \psi_n(s_1; y) & \psi_n(s_2; y) \end{vmatrix} \geq 0 \tag{2.8}$$

for $s_1 < s_2$ and $x < y$.

Letting n go to infinity, (2.8) implies that

$$\begin{vmatrix} \phi(s_1; x) & \phi(s_2; x) \\ \phi(s_1; y) & \phi(s_2; y) \end{vmatrix} \geq 0.$$

If $\phi(s_1; x) > 0$ then $\phi(s_2; x) > 0$, and

$$\frac{\phi(s_2; y)}{\phi(s_2; x)} \geq \frac{\phi(s_1; y)}{\phi(s_1; x)}$$

so ϕ_y/ϕ_x is nondecreasing. Letting $y \to x+$ one concludes that $\psi_x = \phi_{x+}/\phi_x$ is nondecreasing. Also, $\psi_x(t) \leq 1$ and equality holds iff ϕ^t is continuous at x. Consequently, if ϕ^t has a discontinuity at x, then so does ϕ^s for all $s < t$ as long as $\phi^s(x) = \phi_x(s) = \phi(s; x) \neq 0$. ▮

2.11. Corollary. *The set $X = \{x | \phi_x \neq \phi_{x+}\}$ is countable.*

Proof. For every t, ϕ^t has at most a countable number of points of discontinuity, i.e., $X^t = \{x | \phi^t(x) \neq \phi^t(x +)\}$ is countable. It follows from Lemma 2.10 that $X = \bigcup\{X^t | a \leq t < b\} = \bigcup\{X^r | a \leq r < b,\ r \text{ is rational}, \text{ or } r = a\}$, hence the set X is countable. ▮

We now discuss the elements of the cone S for which the measure α has exactly one point of increase. In particular, we study the extreme ray structure of S.

Let $\{\xi_n\}_{n=1}^\infty$ be a sequence of numbers in the interval $[a, b]$. Since both $\{\xi_n\}_{n=1}^\infty$ and $\{\psi_n(\cdot\,; \xi_n)\}_{n=1}^\infty$ are bounded, there exists a subsequence of integers, $\{n_j\}_{j=1}^\infty$ for which $\{\xi_{n_j}\}_{j=1}^\infty$ and $\{\psi_{n_j}(\cdot\,; \xi_{n_j})\}_{j=1}^\infty$ converge. Note that the convergence is in the topology defined by (2.3). In particular it is uniform on every closed subinterval of $[a, b)$. Letting $\underset{\sim}{\xi} = (\{\xi_{n_j}\}_{j=1}^\infty, \{n_j\}_{j=1}^\infty)$, define

$$\lim \underset{\sim}{\xi} = \lim_{j \to \infty} \xi_{n_j}, \tag{2.9}$$

and call $I(\underset{\sim}{\xi}) = \{n_j\}_{j=1}^\infty$ the index set of $\underset{\sim}{\xi}$. Define

$$\phi_{\underset{\sim}{\xi}} = I(\underset{\sim}{\xi}) - \lim \psi_n(\cdot\,; \xi_n). \tag{2.10}$$

For the sake of simplicity we write $\underline{\xi} - \lim \psi_n(\cdot; \xi_n)$ for $I(\underline{\xi}) - \lim \psi_n(\cdot; \xi_n)$.

When $I(\underline{\xi}) = \{n | n \geq n_0\}$ we write $\underline{\xi} = \{\xi_n\}_{n=n_0}^{\infty}$. Clearly, $\phi_{\underline{\xi}}$ belongs to $S \cap B$. Let $\lim \underline{\xi} = x$. If $\xi_n \leq x$ for infinitely many values of n, $n \in I(\underline{\xi})$, then $\phi_y \geq \phi_{\underline{\xi}} \geq \phi_x$ for every y, $y < x$. Letting $y \to x$, the left continuity of ϕ_y (in y) implies that $\phi_{\underline{\xi}} = \phi_x$. Note that if ϕ' is continuous at x, and $\lim \underline{\xi} = x$, then $\phi_{\underline{\xi}}(t) = \phi(t; x) = \phi_x$.

Let $\underline{\xi}$ and $\underline{\eta}$ be two such sequences with limits x and y, respectively. We say that $\underline{\xi} \leq \underline{\eta}$ if $\phi_{\underline{\xi}} \geq \phi_{\underline{\eta}}$. When $\phi_{\underline{\xi}} = \phi_{\underline{\eta}}$ we say that $\underline{\xi}$ and $\underline{\eta}$ are equivalent and write $\underline{\xi} \sim \underline{\eta}$. We say that $\underline{\xi} < \underline{\eta}$ if $\underline{\xi} \leq \underline{\eta}$ and $\underline{\xi} \nsim \underline{\eta}$. In particular, when $\underline{\xi}$ and $\underline{\eta}$ have the same index set I, and $\xi_n \leq \eta_n$ holds for infinitely many values of $n \in I$ then $\underline{\xi} \leq \underline{\eta}$.

We now show that the set Ξ of all sequences $\underline{\xi}$, defined above, is totally ordered.

2.12. LEMMA. *Let $\underline{\xi} \in \Xi$ and let $\phi_{\underline{\xi}}$ be defined by* (2.10). *There exists a sequence $\underline{\xi}' = \{\xi'_n\}_{n=1}^{\infty}$ such that for every $t \in [a, b)$,*

$$\phi_{\underline{\xi}}(t) = \lim_{n \to \infty} \psi_n(t; \xi'_n).$$

Proof. For every $n \in I(\underline{\xi})$ set $\xi'_n = \xi_n$. Let n_j, $n_{j+1} \in I(\underline{\xi})$ and assume that $n_j + 1 < n_{j+1}$. We now define ξ'_n for $n_j < n < n_{j+1}$.

Case A: $\xi_{n_j} \leq \xi_{n_{j+1}}$. Set $\xi'_n = \xi_{n_{j+1}}$ for every $n_j < n < n_{j+1}$. Since $\psi_n(t; x)$ is a nonincreasing function of n and x,

$$\psi_{n_j}(\cdot; \xi_{n_j}) \geq \psi_n(\cdot; \xi'_n) \geq \psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}}). \tag{2.11}$$

Case B: $\xi_{n_{j+1}} < \xi_{n_j}$. Since for all x and t, $\{\psi_n(t; x)\}_{n=0}^{\infty}$ is a nonincreasing sequence, it follows that

$$\psi_n(\cdot; \xi_{n_j}) \leq \psi_{n_j}(\cdot; \xi_{n_j}) \tag{2.12}$$

and

$$\psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}}) \leq \psi_n(\cdot; \xi_{n_{j+1}}) \tag{2.13}$$

for all $n_j < n < n_{j+1}$. By Lemma 1.2, strict inequality holds in (2.12) and (2.13) in (ξ_{n_j}, b) and $(\xi_{n_{j+1}}, b)$, respectively.

Recall from Lemma 1.2 that $\psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}}) - \psi_{n_j}(\cdot; \xi_{n_j})$ has at most one root in (ξ_{n_j}, b). Assume first that the equation

$$\psi_{n_j}(t; \xi_{n_j}) = \psi_{n_{j+1}}(t; \xi_{n_{j+1}}) \tag{2.14}$$

has one root in (ξ_{n_j}, b) and denote it by t_0. By a continuity argument, one can show that there exists ξ_n', $\xi_{n_{j+1}} < \xi_n' < \xi_{n_j}$ such that

$$\psi_n(t_0; \xi_n') = \psi_{n_j}(t_0; \xi_{n_j}) = \psi_{n_{j+1}}(t_0; \xi_{n_{j+1}}).$$

In particular, this follows from (2.12) and (2.13). Moreover, it follows from Lemma 1.2 that the functions $\psi_{n_j}(\cdot; \xi_{n_j}) - \psi_n(\cdot; \xi_{n_j})$, $\psi_n(\cdot; \xi_{n_{j+1}}) - \psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}})$ and $\psi_{n_j}(\cdot; \xi_{n_j}) - \psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}})$ have a sign change at t_0. This implies that for every $t \in [a, b]$,

$$\psi_n(t; \xi_n') \text{ lies between } \psi_{n_j}(t; \xi_{n_j}) \text{ and } \psi_{n_{j+1}}(t; \xi_{n_{j+1}}). \tag{2.15}$$

In case that (2.14) has no roots in (ξ_{n_j}, b), the inequalities

$$\psi_n(\cdot; \xi_{n_j}) \le \psi_{n_j}(\cdot; \xi_{n_j}) \le \psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}}) \le \psi_n(\cdot; \xi_{n_{j+1}})$$

hold for every $n_j < n < n_{j+1}$. We claim that for some $\xi \in [\xi_{n_{j+1}}, \xi_{n_j}]$ we have $\psi_{n_j}(\cdot; \xi_{n_j}) \le \psi_n(\cdot; \xi) \le \psi_{n_{j+1}}(\cdot; \xi_{n_{j+1}})$. Set

$$A = \Big\{\xi | \xi_{n_{j+1}} < \xi < \xi_{n_j}, \exists t = t(\xi), \text{ in } (\xi_{n_{j+1}}, b),$$
$$\text{such that } \psi_n(t; \xi) > \psi_{n_{j+1}}(t; \xi_{n_{j+1}})\Big\}$$

and

$$B = \Big\{\xi | \xi_{n_{j+1}} < \xi < \xi_{n_j}, \exists t = t(\xi), \text{ in } (\xi_{n_j}, b),$$
$$\text{such that } \psi_n(t; \xi) < \psi_{n_j}(t; \xi_{n_j})\Big\}.$$

The continuity of $\psi_n(t; \cdot)$ implies that both A and B are open. Moreover, this continuity together with (2.12) and (2.13) imply that all $\xi \in (\xi_{n_{j+1}}, \xi_{n_{j+1}} + \varepsilon)$ belong to A, and all $\xi \in (\xi_{n_j} - \varepsilon, \xi_{n_j})$ belong to B, for some positive ε, i.e., both A and B are not empty. Next we show that A and B are disjoint. Assume they are not. For $\xi \in A \cap B$ there exist two points $t_1, t_2 \in (\xi_{n_{j+1}}, b)$ such that $\psi_n(t_1; \xi) < \psi_{n_j}(t_1; \xi_{n_j})$ and $\psi_n(t_2; \xi) > \psi_{n_{j+1}}(t_2; \xi_{n_{j+1}})$. If $t_1 > t_2$ then the equation

$$\psi_n(t; \xi) = \psi_{n_{j+1}}(t; \xi_{n_{j+1}})$$

has at least two roots in $(\xi_{n_{j+1}}, b)$, and if $t_1 < t_2$, then the equation

$$\psi_n(t; \xi) = \psi_{n_j}(t; \xi_{n_j})$$

has two roots in (ξ_{n_j}, b), in contradiction to Lemma 1.2.

Since the interval $(\xi_{n_{j+1}}, \xi_{n_j})$ is a connected set, it cannot be the union of A and B, i.e., for every $n_j < n < n_{j+1}$ there exists $\xi'_n \in (\xi_{n_{j+1}}, \xi_{n_j}) \setminus (A \cup B)$. For such ξ'_n,

$$\psi_{n_j}(\cdot\,; \xi_{n_j}) \le \psi_n(\cdot\,; \xi'_n) \le \psi_{n_{j+1}}(\cdot\,; \xi_{n_{j+1}}). \tag{2.16}$$

Since for every $t \in [a,b]$, $\lim_{j\to\infty} \psi_{n_j}(t; \xi_{n_j}) = \phi_{\underline{\xi}}(t)$ and since for every $t \in [a,b]$ and all $n_j < n < n_{j+1}$, $\psi_n(t; \xi'_n)$ is between $\psi_{n_j}(t; \xi_{n_j})$ and $\psi_{n_{j+1}}(t; \xi_{n_{j+1}})$, (see (2.11), (2.15) and (2.16)), it follows that

$$\phi_{\underline{\xi}}(t) = \lim_{n\to\infty} \psi_n(t; \xi'_n) \qquad \text{for every } t \in [a,b).$$

Moreover, the convergence is uniform on every closed subinterval of $[a,b)$. ∎

2.13. COROLLARY. (a) *For every* $\underline{\xi}, \underline{\eta} \in \Xi$, *one of the following holds*: (i) $\underline{\xi} < \underline{\eta}$, (ii) $\underline{\xi} > \underline{\eta}$, *or* (iii) $\underline{\xi} \sim \underline{\eta}$. (b) *If* $\phi_{\underline{\xi}}(t) > \phi_{\underline{\eta}}(t)$ *for some* t, *then* $\underline{\xi} < \underline{\eta}$.

Proof. Let ξ' and η' be defined as in Lemma 2.12. If for almost all n, $\xi'_n < \eta'_n$ ($\xi'_n > \eta'_n$) then, since for all n $\psi_n(t; x)$ is nonincreasing in x, we get $\phi_{\underline{\xi}} \ge \phi_{\underline{\eta}}$ ($\phi_{\underline{\xi}} \le \phi_{\underline{\eta}}$). If this is not the case, then both relations, $\xi'_n \le \eta'_n$ and $\xi'_n \ge \eta'_n$ hold infinitely many times from which one deduces that $\phi_{\underline{\xi}} = \phi_{\underline{\eta}}$. This concludes the proof of part (a). Part (b) follows from part (a). ∎

2.14. LEMMA. *The set* $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ *is compact in the topology defined by the family of seminorms* (2.3). *Moreover, if* $\lim_{m\to\infty} \phi_{\underline{\xi}_m}$ *exists, then there exists* $\underline{\xi}$ *with* $\lim \underline{\xi} = \lim_{j\to\infty} \lim \underline{\xi}_{m_j}$ *for some subsequence of integers* $\{m_j\}_{j=1}^{\infty}$ *and* $\lim_{m\to\infty} \phi_{\underline{\xi}_m} = \phi_{\underline{\xi}}$.

Proof. It is sufficient to show that $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ is sequentially compact. Let $\{\phi_{\underline{\xi}_m}\}_{m=1}^{\infty}$ be a sequence of functions with $\underline{\xi}_m \in \Xi$. By (2.9) and (2.10), there exist sequences

$$\underline{\eta}^{(m)} = \left(\left\{\eta^{(m)}_{n_j(m)}\right\}_{j=1}^{\infty}, \left\{n_j(m)\right\}_{j=1}^{\infty}\right), \qquad m = 1,2,3,\ldots,$$

with $\lim_{j\to\infty} \eta^{(m)}_{n_j(m)} = x_m$ such that $\lim_{j\to\infty} \psi_{n_j(m)}(\cdot\,; \eta^{(m)}_{n_j(m)}) = \phi_{\underline{\xi}_m}$.

We may assume (taking subsequence if necessary) that $\lim_{m\to\infty} x_m = x_0$.

Let $m_1 > 2/(b-a)$ be an integer such that $|x_{m_1} - x_0| < 1/2$ and let $n(m_1) \in I(\underline{\eta}_{m_1})$ be such that

$$\text{(i)}_1\ |\eta^{(m_1)}_{n(m_1)} - x_{m_1}| < 1/2 \quad \text{and} \quad \text{(ii)}_1\ \left\|\psi_{n(m_1)}(\cdot\,; \eta^{(m_1)}_{n(m_1)}) - \phi_{\underline{\xi}_{m_1}}\right\|_{m_1}^{m_1} < 1/2.$$

Suppose $m_1, m_2, \ldots, m_{j-1}$ had been chosen. Choose $m_j > m_{j-1}$ such that $|x_{m_j} - x_0| < 1/2^j$ and let $n(m_j) \in I(\underline{\eta}_{m_j})$ be such that

$$(\mathrm{i})_j\ |\eta^{(m_j)}_{n(m_j)} - x_{m_j}| < 1/2^j \quad \text{and} \quad (\mathrm{ii})_j\ \left\|\psi_{n(m_j)}\left(\cdot\,;\eta^{(m_j)}_{n(m_j)}\right) - \phi_{\underline{\xi}_{m_j}}\right\|^{m_j}_{m_j} < 1/2^j.$$

Clearly, $\lim_{j\to\infty} \eta^{(m_j)}_{n(m_j)} = x_0$. Let $\lim_{j\to\infty} \psi_{n(m_j)}(\cdot\,;\eta^{(m_j)}_{n(m_j)}) = \phi_{\underline{\xi}}$ (taking subsequence if necessary.)

We now show that $\lim_{j\to\infty}\phi_{\underline{\xi}_{m_j}} = \phi_{\underline{\xi}}$. Given $\varepsilon > 0$ and two integers n, k there exists j_0 with $m_{j_0} > \max(n,k)$ such that for all $j > j_0$

$$\left\|\psi_{n(m_j)}\left(\cdot\,;\eta^{(m_j)}_{n(m_j)}\right) - \phi_{\underline{\xi}}\right\|^{n}_{k} < \varepsilon/2$$

and

$$\left\|\psi_{n(m_j)}\left(\cdot\,;\eta^{(m_j)}_{n(m_j)}\right) - \phi_{\underline{\xi}_{m_j}}\right\|^{n}_{k} < \varepsilon/2.$$

Thus for $j > j_0$, $\|\phi_{\underline{\xi}_{m_j}} - \phi_{\underline{\xi}}\|^n_k < \varepsilon$, i.e., $\lim_{j\to\infty}\phi_{\underline{\xi}_{m_j}} = \phi_{\underline{\xi}}$.

In particular, since for all $x \in [a,b)$ ϕ_x belongs to $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ so does ϕ_{x+}. ∎

2.15. Lemma. *Let $\underline{\xi}, \underline{\eta} \in \Xi$. If $\phi_{\underline{\xi}}(t) = \phi_{\underline{\eta}}(t)$ for some $t \in [a,b]$, then either $\phi_{\underline{\xi}}(t) = \phi_{\underline{\eta}}(t) = 0$ or $\phi_{\underline{\xi}}(s) = \phi_{\underline{\eta}}(s)$ for all $s \geq t$.*

Proof. By Corollary 2.13 we may assume that $\underline{\xi} < \underline{\eta}$. Obviously $\phi_{\underline{\xi}} \geq \phi_{\underline{\eta}}$. Assume that $\phi_{\underline{\xi}}(t) \neq 0$. Inequality (2.8), together with Lemma 2.12, implies

$$\begin{vmatrix} \phi_{\underline{\xi}}(t) & \phi_{\underline{\eta}}(t) \\ \phi_{\underline{\xi}}(s) & \phi_{\underline{\eta}}(s) \end{vmatrix} \geq 0 \tag{2.17}$$

for $t < s$.

Since $\phi_{\underline{\xi}}(t) = \phi_{\underline{\eta}}(t) > 0$ one concludes that $\phi_{\underline{\xi}}(s) \leq \phi_{\underline{\eta}}(s)$. This implies that equality holds for all $s \geq t$. ∎

We now show that this family has a mean value property, in particular, the gap between ϕ_x and ϕ_{x+} is filled.

2.16. Proposition. *Let $\underline{\xi} < \underline{\eta}$ be two sequences with limits x_0 and y_0, respectively. If for some $t \in [a,b)$*

$$\phi_{\underline{\eta}}(t) < r < \phi_{\underline{\xi}}(t), \tag{2.18}$$

then there exists a sequence $\underline{\zeta}$, $\underline{\xi} < \underline{\zeta} < \underline{\eta}$ such that $\phi_{\underline{\zeta}}(t) = r$.

Proof. By Lemma 2.12 we may extend the sequences $\underline{\xi}$ and $\underline{\eta}$ to $\{\xi_i'\}_{i=0}^{\infty}$ and $\{\eta_i'\}_{i=0}^{\infty}$ respectively. Assume first that $x_0 > a$ and $y_0 < b$. Since $\underline{\xi} < \underline{\eta}$, we have $x_0 \le y_0$. Let $x < x_0$ and $y > y_0$. There exists n_0 such that for all $k > n_0$, $\xi_k' > x$ and $\eta_k' < y$.

Let $\varepsilon = (1/2)\min(\phi_{\underline{\xi}}(t) - r,\ r - \phi_{\underline{\eta}}(t))$. There exists $n(\varepsilon) > n_0$ such that

$$\psi_k(t; x) \ge \psi_k(t; \xi_k') \ge \phi_{\underline{\xi}}(t) - \varepsilon > r, \qquad k \ge n(\varepsilon), \tag{2.19}$$

and

$$\psi_k(t; y) \le \psi_k(t; \eta_k') \le \phi_{\underline{\eta}}(t) + \varepsilon < r, \qquad k \ge n(\varepsilon). \tag{2.20}$$

The first inequality in each of the formulae (2.19) and (2.20) follows from the monotonicity of $\psi_n(t; \cdot\,)$, the second from the definitions of $\phi_{\underline{\xi}}$ and $\phi_{\underline{\eta}}$, and the third from the definition of ε.

Let $n > n(\varepsilon)$. For $k > n$,

$$\psi_n(t; x) \ge \psi_k(t; x) > r. \tag{2.21}$$

Also,

$$\psi_n(t; y) \le \psi_{n(\varepsilon)}(t; y) < r. \tag{2.22}$$

The first inequality in each of the formulae (2.21) and (2.22) follows from the monotonicity of sequences $\psi_n(t; x)$, and $\psi_n(t; y)$, the second from (2.19) and (2.20).

Thus we conclude that

$$\psi_n(t; y) < r < \psi_n(t; x), \qquad n > n(\varepsilon).$$

This together with the continuity of $\psi_n(t, \cdot\,)$ imply that there exists ζ_n, $x < \zeta_n < y$ such that $\psi_n(t, \zeta_n) = r$, i.e., there exists a sequence $\underline{\zeta}$ with $\lim \underline{\zeta} = z_0$ (taking a subsequence of $\{\zeta_n\}_{n=1}^{\infty}$ if necessary) such that $\phi_{\underline{\zeta}}$ exists and $\phi_{\underline{\zeta}}(t) = r$. Corollary 2.13 implies that $\underline{\xi} < \underline{\zeta} < \underline{\eta}$.

If $x_0 = a$ or $y_0 = b$, the proof is valid with $x = x_0$ and $y = y_0$, respectively. ∎

2.17. Lemma. *Let f be a bounded SGAM function and let α_n and α be defined as in Lemmas 2.5 and 2.6. If x_0 is the only point of increase of α then $f/f(b-\,)$ is in the closed convex hull of the functions $\phi_{\underline{\xi}}$ with $\lim \underline{\xi} = x_0$. In particular, if ϕ^t is continuous at x_0 for every t then $f/f(b-\,) = \phi_{x_0}$.*

Proof. The function f is not identically zero and we may assume that $f(b-\,) = 1$. Thus, $\alpha_n([a, b]) = 1$ for all n.

For every $n, m = 1, 2, 3, \ldots$ Let

$$a = x^n_{m,0} < x^n_{m,1} < \cdots < x^n_{m,m} = b \tag{2.23}$$

be such that $\alpha_n(I^n_{m,k}) = 1/m$, with $I^n_{m,k}$ denoting the interval $[x^n_{m,k}, x^n_{m,k+1})$, $k = 0, 1, \ldots, m-1$. Since by Lemma 2.5, $f(t) = \int_a^b \psi_n(t; x)\, d\alpha_n(x)$, $n = 0, 1, 2, \ldots$, one has

$$\sum_{k=1}^{m} (1/m)\psi_n(t; x^n_{m,k}) \le f(t) \le \sum_{k=0}^{m-1} (1/m)\psi_n(t; x^n_{m,k}). \tag{2.24}$$

Letting n go to infinity (taking subsequences if necessary), one gets

$$A_m(t) \le f(t) \le B_m(t), \tag{2.25}$$

where

$$\begin{aligned} A_m(t) &= \sum_{k=1}^{m} (1/m)\phi_{\underline{x}_{m,k}}(t) \\ B_m(t) &= \sum_{k=0}^{m-1} (1/m)\phi_{\underline{x}_{m,k}}(t) \end{aligned} \tag{2.26}$$

with $\underline{x}_{m,k} = (\{x^{n_j(m,k)}_{m,k}\}_{j=1}^{\infty}, \{n_j(m,k)\}_{j=1}^{\infty})$, $k = 0, 1, \ldots, m$, $m = 1, 2, 3, \ldots$ defined by (2.23). Since for every m there is a finite number of sequences we may assume that $n_j(m,k) = n_j(m)$, $k = 1, 2, \ldots, m$, $j = 1, 2, 3, \ldots$, namely,

$$\underline{x}_{m,k} = \left(\left\{x^{n_j(m)}_{m,k}\right\}_{j=1}^{\infty}, \left\{n_j(m)\right\}_{j=1}^{\infty}\right), \qquad k = 0, 1, \ldots, m,\ m = 1, 2, 3, \ldots \tag{2.27}$$

From (2.25) and (2.26), it follows that

$$0 \le B_m(t) - A_m(t) = (1/m)(\phi_{\underline{a}}(t) - \phi_{\underline{b}}(t)), \tag{2.28}$$

$\underline{a}$ and $\underline{b}$ being the constant sequences $\{a, a, a, \ldots\}$, and $\{b, b, b, \ldots\}$, respectively.

Since for every closed interval $I \subset [a, b]$, $\alpha_n(I)$ tends to 0 if $x_0 \notin I$, we have

$$\lim \underline{x}_{m,k} = \begin{cases} x_0, & 0 < k < m, \\ a, & k = 0, \\ b, & k = m. \end{cases} \tag{2.29}$$

Now, (2.25) and (2.28) imply that $\lim_{m\to\infty}\{(1/2)(A_m(t)+B_m(t))\} = f(t)$, i.e.,

$$\lim_{m\to\infty}\left\{\sum_{k=1}^{m-1}(1/m)\phi_{\underline{x}_{m,k}}(t)+(1/2m)(\phi_{\underline{a}}(t)+\phi_{\underline{b}}(t))\right\}=f(t).$$

From here we conclude that

$$\lim_{m\to\infty}\left\{\sum_{k=1}^{m-1}(1/m)\phi_{\underline{x}_{m,k}}(t)\right\}=f(t),$$

and hence

$$\lim_{m\to\infty}\left\{\sum_{k=1}^{m-1}(1/(m-1))\phi_{\underline{x}_{m,k}}(t)\right\}=f(t). \tag{2.30}$$

The convergence is uniform on every closed subinterval of $[a,b)$. Since $\{f\in C_A|\lim_{t\to b}f(t)\le 1\}$ is a compact set in the generalized $C^\infty(a,b)$ topology, a subsequence of $\{\sum_{k=1}^{m-1}(1/(m-1))\phi_{\underline{x}_{m,k}}\}$ converges to f in this topology, i.e., f is in the closed convex hull of the set

$$\{\phi_{\underline{\xi}}|\lim\underline{\xi}=x_0\}.$$

In the case that ϕ' is continuous at x_0 for all t, $\phi_{\underline{\xi}}=\phi_{x_0}$, whence $f=\phi_{x_0}$. ∎

2.18. LEMMA. *Let $\phi_{\underline{x}_{m,k}}$, $k=1,2,\dots,m-1$, $m=1,2,\dots$ be as in Lemma 2.17, and let $f = M - \lim\{\sum_{k=1}^{m-1}(1/(m-1))\phi_{\underline{x}_{m,k}}\}$ for some sequence of integers M. If f is not a positive multiple of any $\phi_{\underline{\eta}}$, then f does not generate an extreme ray in S.*

Proof. For every $m\in M$ set

$$f_m=\sum_{k=1}^{m-1}(1/(m-1))\phi_{\underline{x}_{m,k}}. \tag{2.31}$$

It follows from (2.23) and (2.27) that

$$\phi_{\underline{x}_{m,0}}\ge\phi_{\underline{x}_{m,1}}\ge\cdots\ge\phi_{\underline{x}_{m,m}}. \tag{2.32}$$

Also, each of these functions is nonnegative and bounded by 1.

We may assume that f is not identically zero, i.e., there exists $t_0\in(a,b)$ such that $f(t_0)=p>0$. There exists $m_0=m_0(t_0)$ such that $f_m(t_0)>(3/4)p$ for all $m>m_0$, $m\in M$. We may assume that $m_0>1+4/p$.

Since each of the summands in (2.31) is nonnegative and bounded by $1/(m-1)$, there exists $m' = m'(m)$ such that

$$p/4 \leq \sum_{k=1}^{m'} (1/(m-1))\phi_{\underline{x}_{m,k}}(t_0) \leq p/2.$$

Define $g_m = \sum_{k=1}^{m'}(1/(m-1))\phi_{\underline{x}_{m,k}}$.

Setting $h_m = f_m - g_m$, it follows that $h_m(t_0) \geq p/4$.

Let $\underline{\eta}_m \in \Xi$ be such that

$$\phi_{\underline{x}_{m,m'}} \geq \phi_{\underline{\eta}_m} \geq \phi_{\underline{x}_{m,m'+1}}. \tag{2.33}$$

Applying (2.17) and the inequalities (2.32) and (2.33), one gets

$$\begin{vmatrix} \phi_{\underline{x}_{m,k}}(t) & \phi_{\underline{\eta}_m}(t) \\ \phi_{\underline{x}_{m,k}}(s) & \phi_{\underline{\eta}_m}(s) \end{vmatrix} \geq 0,$$

for all $k = 1, 2, \ldots, m'$, and all $t < s$. From the linearity of the determinant in the first column it follows that

$$\begin{vmatrix} g_m(t) & \phi_{\underline{\eta}_m}(t) \\ g_m(s) & \phi_{\underline{\eta}_m}(s) \end{vmatrix} \geq 0 \tag{2.34}$$

for all $t < s$.

Similarly,

$$\begin{vmatrix} \phi_{\underline{\eta}_m}(t) & h_m(t) \\ \phi_{\underline{\eta}_m}(s) & h_m(s) \end{vmatrix} \geq 0 \tag{2.35}$$

for all $t < s$.

Letting $m \to \infty$ (taking subsequences if necessary) and applying Lemma 2.14, one sees that the functions g_m, h_m and $\phi_{\underline{\eta}_m}$ converge, in the topology defined by (2.3), to g, h, and $\phi_{\underline{\eta}}$, respectively. The inequalities (2.34) and (2.35) imply

$$\begin{vmatrix} g(t) & \phi_{\underline{\eta}}(t) \\ g(s) & \phi_{\underline{\eta}}(s) \end{vmatrix} \geq 0 \tag{2.36}$$

and

$$\begin{vmatrix} \phi_{\underline{\eta}}(t) & h(t) \\ \phi_{\underline{\eta}}(s) & h(s) \end{vmatrix} \geq 0 \tag{2.37}$$

for all $t < s$.

Clearly

$$f = g + h \tag{2.38}$$

and g and $h \neq 0$. Next we show that $\phi_{\underline{\eta}} \neq 0$.

It follows from (2.32) and (2.33) that for every m,

$$\phi_{\underline{\eta}_m} > \phi_{\underline{x}_{m,m'+1}} \geq \sum_{k=m'+1}^{m} (1/(m-m'))\phi_{\underline{x}_{m,k}}$$

$$\geq \sum_{k=m'+1}^{m} (1/(m-1))\phi_{\underline{x}_{m,k}} = h_m.$$

Letting m go to ∞, one concludes that $\phi_{\underline{\eta}}(t_0) \geq h(t_0) \geq p/4 > 0$.

We now show that g and h do not belong to the same ray of S. Assume to the contrary that they do belong to the same ray. In this case equality holds in both (2.36) and (2.37) for all t and s, $t < s$. This implies that both g and h are positive multiples of $\phi_{\underline{\eta}}$. From (2.38) it follows that f is a positive multiple of $\phi_{\underline{\eta}}$, in contradiction to the assumptions of the lemma. ∎

3. The Extreme Ray Representation

We now state conditions under which every $\phi_{\underline{\xi}}$, not identically zero, generates an extreme ray of S.

3.1. Definition. Let $\underline{\xi}$ and $\phi_{\underline{\xi}}$ be as above and let

$$t_{\underline{\xi}} = \sup\{t \mid \phi_{\underline{\xi}}(t) = 0\}.$$

We say that the function $\phi_{\underline{\xi}}$, not identically equal to zero, has property (*) if for every $\underline{\eta} \in \Xi$, $\underline{\eta} < \underline{\xi}$

$$\lim_{t \to t_{\underline{\eta}}} \frac{\phi_{\underline{\xi}}(t)}{\phi_{\underline{\eta}}(t)} = 0$$

We say that the family $\{\phi_{\underline{\xi}} \mid \underline{\xi} \in \Xi,\ \phi_{\underline{\xi}}(b-) \neq 0\}$ has property (*) if each of its elements has property (*).

Letting $[\underline{\xi}] = \{\underline{\eta} \mid \underline{\eta} \sim \underline{\xi}\}$ denote the equivalence class of the sequence $\underline{\xi}$ we put

$$\phi_{[\underline{\xi}]} = \phi_{\underline{\xi}}. \tag{3.1}$$

3.2. THEOREM. *Let the cone S and the family* $\{\phi_{[\underline{\xi}]} | \underline{\xi} \in \Xi, \phi_{\underline{\xi}}(b-) \neq 0\}$ *be defined as above. All extreme rays are generated by elements of this family. If* $\phi_{\underline{\xi}_0}$ *has property* (*) *then* $\phi_{[\underline{\xi}_0]}$ *generates an extreme ray of* $S \cap B$.

Proof. The first claim follows from Lemmas 2.17 and 2.18. Let $S_0 = \{f | f \in S, f(b-) = 1\}$ and let $\phi_{[\underline{\xi}_0]} \in \{\phi_{[\underline{\xi}]} | \underline{\xi} \in \Xi, \phi_{[\underline{\xi}]}(b-) \neq 0\}$. Since S_0 is compact and convex, and the family $\{\varphi_{[\underline{\xi}]} | \underline{\xi} \in \Xi, \phi_{[\underline{\xi}]}(b-) \neq 0\}$, where $\varphi_{[\underline{\xi}]} = \phi_{[\underline{\xi}]}/\phi_{[\underline{\xi}]}(b-)$, contains all its extreme points, the well known theorem of Choquet (see, e.g., [5]) implies that every $f \in S_0$ admits a representation $L(f) = \int L\, d\lambda_f$ for every continuous linear functional L, where λ_f is supported by the set of extreme points of S_0. Since the set $\{\varphi_{[\underline{\xi}]} | \underline{\xi} \in \Xi, \phi_{[\underline{\xi}]}(b-) \neq 0\}$, contains all extreme points of S_0, and since there is a one-to-one correspondence, $T: [\underline{\xi}] \to \varphi_{[\underline{\xi}]}$, between this set and the set $\{[\underline{\xi}] | \underline{\xi} \in \Xi, \phi_{[\underline{\xi}]}(b-) \neq 0\}$, we have

$$L(f) = \int L(\varphi_{[\underline{\xi}]})\, d\nu_f([\underline{\xi}]),$$

where $\nu_f = \lambda_f \circ T^{-1}$. For the "point evaluation" functionals we have the following representation:

$$f(t) = \int \varphi_{[\underline{\xi}]}(t)\, d\nu_f([\underline{\xi}]), \qquad \text{for every } t \in [a, b). \tag{3.2}$$

In particular,

$$\varphi_{[\underline{\xi}_0]}(t) = \int \varphi_{[\underline{\xi}]}(t)\, d\nu_{\varphi_{[\underline{\xi}_0]}}([\underline{\xi}]), \qquad \text{for every } t \in [a, b)$$

Let $t > t_{\underline{\xi}_0}$ and let $\underline{\xi}_1 < \underline{\xi}_0$; then

$$1 = \int_{\underline{\xi} < \underline{\xi}_1} \frac{\varphi_{[\underline{\xi}]}(t)}{\varphi_{[\underline{\xi}_0]}(t)}\, d\nu_{\varphi_{[\underline{\xi}_0]}}([\underline{\xi}]) + \int_{\underline{\xi}_1 \leq \underline{\xi}} \frac{\varphi_{[\underline{\xi}]}(t)}{\varphi_{[\underline{\xi}_0]}(t)}\, d\nu_{\varphi_{[\underline{\xi}_0]}}([\underline{\xi}]).$$

Letting $t \to t_{\underline{\xi}}+$, property (*) implies that the integrand of the first integral tends to infinity, hence the measure $\nu_{\varphi_{[\underline{\xi}_0]}}$ must vanish on the set $\{[\underline{\xi}] | \underline{\xi} < \underline{\xi}_1\}$ i.e., $\nu_{\varphi_{[\underline{\xi}_0]}}$ is supported by the set $\{[\underline{\xi}] | \underline{\xi} \geq \underline{\xi}_1\}$. Since this holds for every $\underline{\xi}_1$ with $\underline{\xi}_1 < \underline{\xi}_0$, it follows that $\nu_{\varphi_{[\underline{\xi}_0]}}$ is supported by the set $\{[\underline{\xi}] | \underline{\xi} \geq \underline{\xi}_0\}$. It follows from (2.17) (letting $s \to b-$) that

$$\varphi_{[\underline{\xi}_0]}(t) \geq \varphi_{[\underline{\xi}]}(t). \tag{3.3}$$

For every $\underline{\eta}$, $\underline{\eta} > \underline{\xi}_0$, there exists $t = t(\underline{\xi}_0, \underline{\eta})$ such that for all $\underline{\xi} \geq \underline{\eta}$ strict inequality holds in (3.3). This implies that $\nu_{\varphi_{[\underline{\xi}_0]}}$ vanishes on the set

$\{[\underline{\xi}] | \underline{\xi} \geq \underline{\eta}\}$. Since this is true for every $\underline{\eta} > \underline{\xi}_0$, $\nu_{\varphi_{[\xi_0]}}$ is supported by the set $\{[\underline{\xi}_0]\}$, i.e., $\varphi_{[\xi_0]}$ is an extreme point of S_0 and hence $\phi_{[\xi_0]}$ generate an extreme ray of S. ▮

Combining (2.1), (2.2), and (3.2), we have:

3.3. THEOREM. *Let* $\{u_i\}_{i=0}^{\infty}$ *be defined by* (1.1)–(1.3) *and let* $\{\phi_{[\underline{\xi}]} | \underline{\xi} \in \Xi\}$ *be defined by* (2.10) *and* (3.1), *then every* $f \in B \cap C_A$ *admits the representation*

$$f = \sum_{i=0}^{\infty} a_i u_i + \int \phi_{[\underline{\xi}]} \, d\mu_f([\underline{\xi}]), \tag{3.4}$$

for some nonnegative measure μ_f.

Moreover, in case that the family $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ *has property* (*), *then* (3.4) *is an extreme ray representation of* f.

In the following theorems we consider the extreme ray structure of the cone S in a special case of SGAM functions.

3.4. THEOREM. *Let* $\{u_i\}_{i=0}^{\infty}$ *and* $\{\phi_{[\underline{\xi}]} | \underline{\xi} \in \Xi\}$ *be defined as above and assume that* (1.6′) *does not hold. A necessary and sufficient conditions that* $\lim \underline{\xi} = a$ *for all* $\underline{\xi} \in \Xi$ *with* $\phi_{\underline{\xi}} \neq 0$ *is that for all* t *and* x, $b > t > x > a$,

$$\phi_x(t) = \phi(t, x) = 0. \tag{3.5}$$

Proof. Let $\underline{\xi} \in \Xi$. Assume that $\lim \underline{\xi} = z > a$. Let $a < x < z$. Then $\phi_x \geq \phi_{\underline{\xi}}$ and the sufficiency of (3.5) follows. Let $\underline{x} = \{x, x, x, \dots\}$. Since $\underline{x} \in \Xi$, and $\phi_{\underline{x}} = \lim_{n \to \infty} (\phi_n(\cdot\,; x)/\phi_n(b; x)) = \phi_x$, (3.5) is necessary as well. ▮

3.5. THEOREM. *Let* $\{u_i\}_{i=0}^{\infty}$ *and* $\{\phi_{[\underline{\xi}]} | \underline{\xi} \in \Xi\}$ *be defined as above and assume that* (1.6′) *does not hold. For* $i = 0, 1, \dots,$ *let*

$$0 < m_i(x, y) = \min\{w_i(t) | x \leq t \leq y\} \leq \max\{w_i(t) | x \leq t \leq y\} = M_i(x, y).$$

If for every c, $a < c < b$ *there exists an* $\varepsilon = \varepsilon(c)$, $\varepsilon > 0$, *such that*

$$\lim_{n \to \infty} \left[\prod_{i=0}^{n} (M_i(c, b)/m_i(c, b)) \right] \varepsilon^n = 0, \tag{3.6}$$

then $\lim \underline{\xi} = a$ *for every* $\underline{\xi}$ *with* $\phi_{\underline{\xi}} \neq 0$.

Proof. Applying Theorem 8.1 of [2, p. 432] to the convexity cone

$$C_{-1}(c, b) \cap \left[\bigcap_{n=0}^{\infty} C(\phi_0(\cdot\,; c) | [c, b], \phi_1(\cdot\,; c) | [c, b], \dots, \phi_n(\cdot\,; c) | [c, b]) \right],$$

where $C_{-1}(c, b)$ is the cone of nonnegative functions on (c, b), one sees that (3.5) holds with $x = c$ (see [1]), hence it holds for all $x \geq c$. Since this is true for all $c > a$, (3.5) is satisfied. Theorem 3.4 implies that $\lim \underline{\xi} = a$ for every $\underline{\xi} \in \Xi$ for which $\phi_{\underline{\xi}} \neq 0$. ▌

4. Example

We now show that there exists a (nontrivial) cone of SGAM functions such that the set $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ has property (*).

We start with the following:

Let $\tilde{w}_0 = 1$ and $\tilde{w}_n(t) = 1/t$, $0 < t \leq 1$, $n \geq 1$. One can show that

$$\tilde{\phi}_0(t; x) = \begin{cases} 0, & t < x, \\ 1, & t \geq x, \end{cases}$$

and for $n \geq 1$

$$\tilde{\phi}_n(t; x) = \begin{cases} 0, & t < x, \\ (1/n!)(\log t - \log x)^n, & t \geq x. \end{cases}$$

Since

$$\frac{\tilde{\phi}_n(t; x)}{\tilde{\phi}_n(1; x)} = \left(1 - \frac{\log t}{\log x}\right)^n, \qquad \text{for } 0 < x \leq t \leq 1,$$

it follows that

$$\lim_{n \to \infty} \frac{\tilde{\phi}_n(t; x)}{\tilde{\phi}_n(1; x)} = 0, \qquad \text{for all } 0 \leq t \leq 1 \text{ and } 0 < x < 1.$$

Let

$$\underline{\xi} = \{\xi_n\}_{n=0}^{\infty} = \{e^{-(n/s)}\}_{n=0}^{\infty}, \qquad s > 0, \tag{4.1}$$

$$\lim_{n \to \infty} \frac{\tilde{\phi}_n(t; \xi_n)}{\tilde{\phi}_n(1; \xi_n)} = \tilde{\phi}_{\underline{\xi}}(t) = t^s. \tag{4.2}$$

We now show that

$$\left\{\tilde{\phi}_{[\underline{\xi}]} | \underline{\xi} = \{e^{-(n/s)}\}_{n=0}^{\infty}, s > 0\right\} = \left\{\tilde{\phi}_{[\underline{\xi}]} | \underline{\xi} \in \Xi\right\}.$$

For every $s > 0$, let f_s be defined by $f_s(t) = t^s$. For every pair (x, y) in the open unit square there exists a number s such that $f_s(x) = y$. Assume that there exists a function $f \notin \{f_s | s > 0\}$ generating an extreme ray in the cone S. There exist two numbers $s_1 \neq s_2$ and two points $t_1, t_2 \in (0,1)$ such that $f_{s_1}(t_1) = f(t_1)$ and $f_{s_2}(t_2) = f(t_2)$. It follows from Lemma 2.15 that f agrees with f_{s_1} on $[t_1, 1)$ and with f_{s_2} on $[t_2, 1)$, i.e., $f_{s_1} = f_{s_2}$.

Note that although $\{\tilde{w}_n\}_{n=0}^{\infty}$ are not $C^{\infty}[0,1]$-functions, Lemma 2.15 is still applicable. We now perturb the functions, $\{\tilde{w}_n\}$, $i = 1, 2, \ldots$ to obtain the desired example.

For $n = 0, 1, 2, \ldots$ let Ω_n be $C^{\infty}[0,1]$ functions satisfying

$$\Omega_n(t) = \begin{cases} 0, & 0 \le t \le \varepsilon_{n+1}, \\ 1, & \varepsilon_n \le t < 1, \end{cases}$$

and $0 < \Omega_n(t) < 1$ and increasing for $\varepsilon_{n+1} < t < \varepsilon_n$, where $\varepsilon_n = e^{-n^2}$.

Define $w_0 = \tilde{w}_0$ and for $n > 1$ set

$$w_n(t) = 1 + (\tilde{w}_n(t) - 1)\Omega_n(t), \qquad 0 < t \le 1,$$

and $w_n(0) = 1$.

Clearly w_n are positive $C^{\infty}[0,1]$ functions, $w_n(t) = \tilde{w}_n(t)$ for $\varepsilon_n \le t \le 1$ and $w_n(t) \le \tilde{w}_n(t)$ for $0 \le t < \varepsilon_n$. Also, it is easy to show that $(w_n/\tilde{w}_n)' \ge 0$.

For every $s > 0$ let $\underline{\xi} = \underline{\xi}(s)$ be a subsequence of (4.1) such that $\underline{\xi} - \lim(\phi_n(\cdot\,; \xi_n)/\phi_n(1; \xi_n))$ exists, and denote the limit by $\phi_{\underline{\xi}}$. Also, $\underline{\xi} - \lim(L_k\phi_n(\cdot\,; \xi_n)/\phi_n(1; \xi_n)) = L_k\phi_{\underline{\xi}}$.

For every s there exists $k(s)$ such that for $k > k(s)$, $\xi_k > \varepsilon_k$. This implies that for $k > k(s)$

$$\frac{L_k\phi_n(\cdot\,; \xi_n)}{\phi_n(1; \xi_n)} = \frac{\tilde{L}_k\tilde{\phi}_n(\cdot\,; \xi_n)}{\phi_n(1; \xi_n)} = \frac{\tilde{L}_k\tilde{\phi}_n(\cdot\,; \xi_n)}{\tilde{\phi}_n(1; \xi_n)}\,\frac{\tilde{\phi}_n(1; \xi_n)}{\phi_n(1; \xi_n)}, \tag{4.3}$$

where L_k and $\tilde{L}_k$ are the operators defined in Section 1 with respect to $\{w_i\}_{i=0}^{\infty}$ and $\{\tilde{w}_i\}_{i=0}^{\infty}$.

Since $\underline{\xi} - \lim(L_k\phi_n(\cdot\,; \xi_n)/\phi_n(1; \xi_n))$ and $\underline{\xi} - \lim(\tilde{L}_k\tilde{\phi}_n(\cdot\,; \xi_n)/\tilde{\phi}_n(1; \xi_n))$ exist and the latter is positive on $(0, 1]$, $\underline{\xi} - \lim(\tilde{\phi}_n(1; \xi_n)/\phi_n(1; \xi_n))$ exists and is positive. Moreover it is ≥ 1, i.e., there exists a constant $a(\underline{\xi}) \ge 1$ such that $L_k\phi_{\underline{\xi}} = a(\underline{\xi})\tilde{L}_k\tilde{\phi}_{\underline{\xi}}$.

Next we show that $\tilde{\phi}_n(1; \cdot\,)/\phi_n(1; \cdot\,)$ is nonincreasing. The claim is clear for $n = 0$. Assume that it is true for $n - 1$. By differentiating, we get

$$\frac{d}{dx}\left(\frac{\tilde{\phi}_n(1; x)}{\phi_n(1; x)}\right) = \frac{w_n(x)\phi_{n-1}(1; x)}{\phi_n(1, x)}\left(\frac{\tilde{\phi}_n(1, x)}{\phi_n(1, x)} - \frac{\tilde{w}_n(x)\tilde{\phi}_{n-1}(1; x)}{w_n(x)\phi_{n-1}(1; x)}\right). \tag{4.4}$$

The right-hand side of (4.4) is nonpositive since

$$\frac{\tilde{\phi}_n(1;x)}{\phi_n(1;x)} = \frac{\tilde{\phi}_n(1;x) - \tilde{\phi}_n(1;1)}{\phi_n(1;x) - \phi_n(1;1)} = \frac{\tilde{w}_n(y)\tilde{\phi}_{n-1}(1;y)}{w_n(y)\phi_{n-1}(1;y)} \le \frac{\tilde{w}_n(x)\tilde{\phi}_{n-1}(1;x)}{w_n(x)\phi_{n-1}(1;x)}. \tag{4.5}$$

The second equality follows from (1.3) and the mean value theorem (for some y, $x < y < 1$). The inequality follows from the monotonicity of $\tilde{w}_n/w_n$ and from the induction assumption.

Combining (4.4) and (4.5), one concludes that $\tilde{\phi}_n(1;\cdot)/\phi_n(1;\cdot)$ is nonincreasing.

If $\lim \underline{\xi} > 0$ then (4.3) is applicable and since $\tilde{\phi}_n(1;\cdot)/\phi_n(1;\cdot)$ is nonincreasing $\underline{\xi} - \limsup \tilde{\phi}_n(1;\xi_n)/\phi_n(1;\xi_n)$ is finite and positive. Since for such $\underline{\xi}$, $\tilde{\phi}_{\underline{\xi}} = 0$ it follows that $\phi_{\underline{\xi}} = 0$.

Finally, let $\underline{\eta} < \underline{\xi}$ with $\lim \underline{\xi} = 0$. By L'Hospital's rule one has

$$\lim_{t\to 0+} \frac{\phi_{\underline{\xi}}(t)}{\phi_{\underline{\eta}}(t)} = \lim_{t\to 0+} \frac{L_k\phi_{\underline{\xi}}(t)}{L_k\phi_{\underline{\eta}}(t)}$$

$$= \frac{a(\underline{\xi})}{a(\underline{\eta})} \lim_{t\to 0+} \frac{\tilde{L}_k\tilde{\phi}_{\underline{\xi}}(t)}{\tilde{L}_k\tilde{\phi}_{\underline{\eta}}(t)} = \frac{a(\underline{\xi})}{a(\underline{\eta})} \lim_{t\to 0+} \frac{\tilde{\phi}_{\underline{\xi}}(t)}{\tilde{\phi}_{\underline{\eta}}(t)},$$

for large k.

Since $\{\tilde{\phi}_{\underline{\xi}} | \underline{\xi} \in \Xi\}$ has property (*), so does $\{\phi_{\underline{\xi}} | \underline{\xi} \in \Xi\}$.

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